Solving Systems Using Matrices

A **matrix** is a rectangular array of numbers.

$$\begin{pmatrix} 3 & 4 \\ -1 & 6 \end{pmatrix}_{2\times 2} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ -3 & 5 & -2 \end{pmatrix}_{3\times 3} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}_{2\times 4} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A **vector** is a matrix with one column.

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix}_{2\times 1} \quad \begin{pmatrix} 5 \\ 10 \\ 20 \end{pmatrix}_{3\times 1} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}_{5\times 1} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$$

A **square** matrix is one having the same number of rows and columns.

The **identity** matrix is a square matrix with 1s in the main diagonal and 0s everywhere else.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots \quad I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n\times n}$$

Augmented Matrix

Consider the following system.

$$\begin{aligned} x - 3y &= 25 \\ 5x + 4y &= 30 \end{aligned}$$

The augmented matrix of this system is

$$\left[\begin{array}{cc|c} 1 & -3 & 25 \\ 5 & 4 & 30 \end{array}\right]$$

and $\begin{bmatrix} 1 & -3 \\ 5 & 4 \end{bmatrix}$ is called the **coefficient matrix** of the system.

For the system

$$\begin{aligned} x_1 - 3x_3 + x_4 &= 1 \\ x_2 - 5x_3 - 10x_4 &= 10 \\ 2x_1 + 4x_2 + 6x_4 &= -8 \\ 4x_3 - 2x_4 &= 19 \end{aligned},$$

the augmented matrix is

$$\begin{bmatrix} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & -5 & -10 & 10 \\ 2 & 4 & 0 & 6 & -8 \\ 0 & 0 & 4 & -2 & 19 \end{bmatrix}$$

Definition

A matrix is in **reduced row echelon form (RREF)** if the following criteria are met:

1. All zero rows are on the bottom of the matrix.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. The leading entry in each nonzero row is 1 (**pivot**).
4. Each leading 1 is the only nonzero entry in its column.

Examples

The matrices $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 7 & 8 \end{bmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ are in RREF,

but the matrices $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & -1 & 7 & 8 \end{bmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ are not.

Elementary Row Operations

When dealing with the augmented matrix of a system of linear equations, the following are permissible operations that do not affect the solution of the system.

1. We can interchange any two rows of the augmented matrix.
2. We can multiply (divide) any row by any nonzero constant.
3. We can add any multiple of any row into any other row.

<u>Note:</u> When solving systems of linear equations using matrices,

a. Convert the system to its augmented matrix.
b. Use elementary row operations to transform the augmented matrix to its RREF.
c. Determine the solution(s) of the system, if consistent.

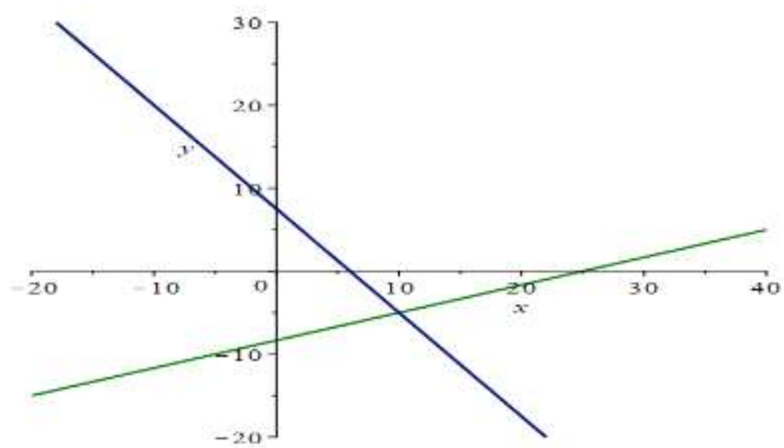
Examples

1. Use matrices to solve the system

$$\begin{aligned} x - 3y &= 25 \\ 5x + 4y &= 30 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & 25 \\ 5 & 4 & 30 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 25 \\ 0 & 19 & -95 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 25 \\ 0 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -5 \end{bmatrix}$$

Solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$

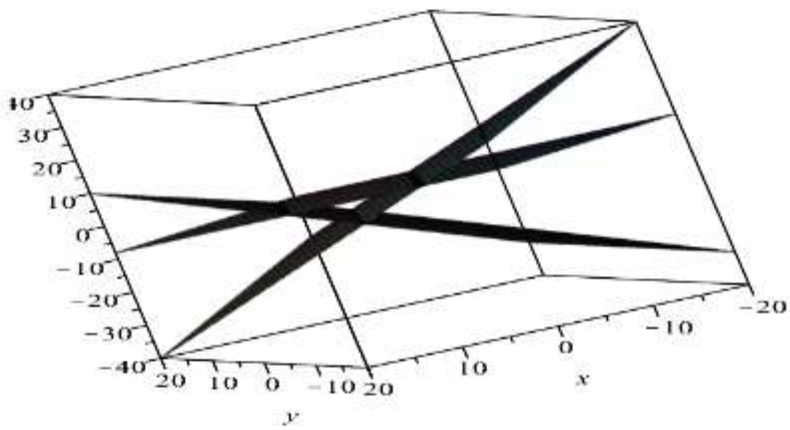


2. Show that the following system has no solution.

$$\begin{aligned} x + y + z &= 0 \\ y + 2z &= 4 \\ x \quad - \quad z &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & -1 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & -1 & -2 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, we have an inconsistent system (no solution).

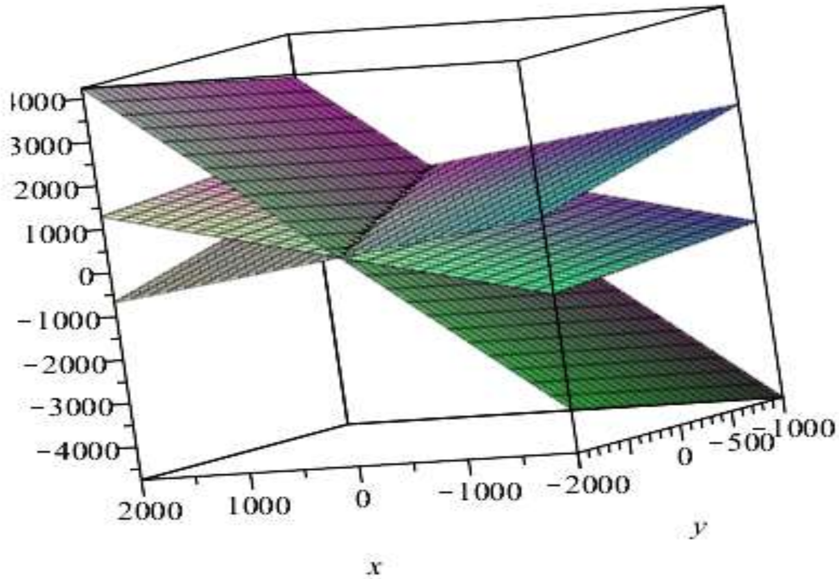


3. Show that the following system has infinitely many solutions.

$$\begin{aligned} x + y + z &= 100 \\ 5x + 10y + 20z &= 2000 \\ -5x \quad + \quad 10z &= 1000 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 100 \\ 5 & 10 & 20 & 2000 \\ -5 & 0 & 10 & 1000 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 100 \\ 0 & 5 & 15 & 1500 \\ 0 & 5 & 15 & 1500 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 100 \\ 0 & 1 & 3 & 300 \\ 0 & 5 & 15 & 1500 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -200 \\ 0 & 1 & 3 & 300 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solutions: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z - 200 \\ 300 - 3z \\ z \end{pmatrix}$



Applications

1. Four hundred people attended a show and the total revenue was $7,550. Adults paid $25 per ticket, while children paid $15.

   a. Translate the information into a system of two linear equations in two variables.

   Let a be the number of adults and c be the number of children. Then

$$\begin{aligned} a + c &= 400 \\ 25a + 15c &= 7550 \end{aligned}$$

   b. Solve the system using any technique.

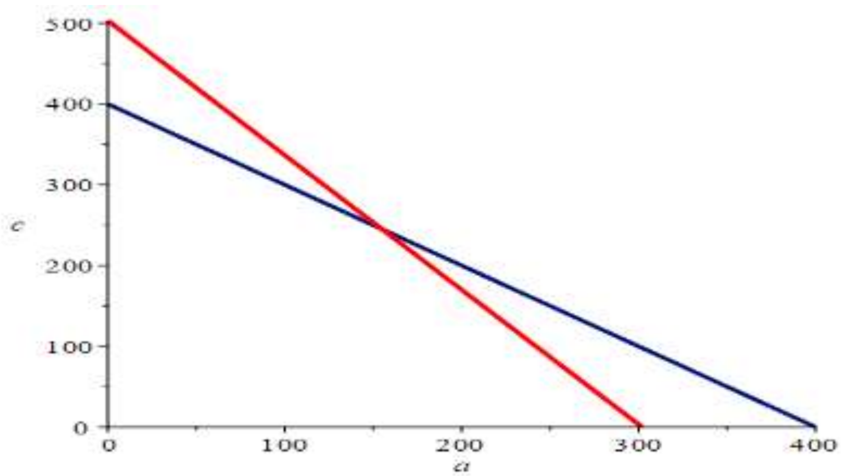
$$\begin{bmatrix} 1 & 1 & 400 \\ 25 & 15 & 7550 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 400 \\ 0 & -10 & -2450 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 400 \\ 0 & 1 & 245 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 155 \\ 0 & 1 & 245 \end{bmatrix}$$

   c. Find how many adults and how many children attended the show.

Solution: $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 155 \\ 245 \end{pmatrix}$

d. Graph the two equations on the same set of axes. Explain what you see from a geometric point of view.

Geometry: The two lines intersect at the point (155, 245).



e. Think about the possible scenario of both adults and children paying the same amount for a ticket.

2. The following system represents currents flowing through a circuit.

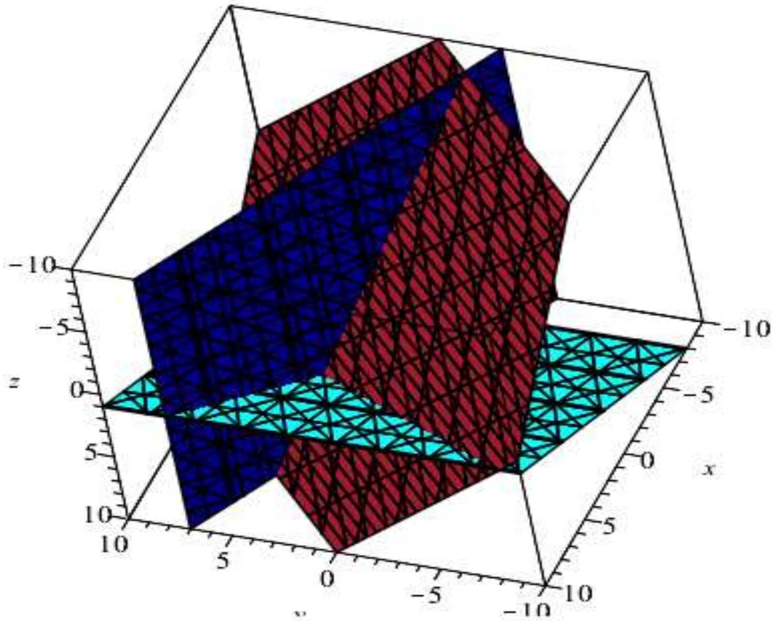
$$\begin{aligned} I_1 + \quad\quad 2I_3 &= 12 \\ -I_1 + 2I_2 \quad\quad &= 4 \\ I_1 + I_2 - I_3 &= 0 \end{aligned}$$

Find the three currents I_1, I_2 and I_3.

$$\begin{bmatrix} 1 & 0 & 2 & 12 \\ -1 & 2 & 0 & 4 \\ 1 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 12 \\ 0 & 2 & 2 & 16 \\ 0 & 1 & -3 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 12 \\ 0 & 1 & 1 & 8 \\ 0 & 1 & -3 & -12 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 2 & 12 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & -4 & -20 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 12 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Solution: $I_1 = 2A$, $I_2 = 3A$, $I_3 = 5A$



Curve Fitting

3. The Hubble space Telescope was deployed on April 24, 1990, by the space shuttle Discovery. Four readings were obtained about the velocity of the shuttle before the solid rocket boosters were jettisoned at $t = 126$ seconds.

t (s)	v (ft/s)
0	0
3	70.05
16	359.98
100	2760.1

a. Find a **cubic** polynomial that fits the above data by setting up and solving a system of linear equations.

b. Use the cubic polynomial to approximate the velocity of the shuttle at $t = 126$ s.

c. What would happen if you attempt to fit a polynomial of higher degree into the same data?

Setting up the equations:

$$V(t) = at^3 + bt^2 + ct + d$$

at (0, 0) $\quad 0 = a(0)^3 + b(0)^2 + c(0) + d$ and $\quad d = 0$

at (3, 70.05) $\quad 70.05 = a(3)^3 + b(3)^2 + c(3) + d$, therefore,

$$27a + 9b + 3c = 70.05$$

at (16, 359.98) $\quad 359.98 = a(16)^3 + b(16)^2 + c(16)$, therefore,

$$4096a + 256b + 16c = 359.98$$

at (100, 2760.1) $\quad 2760.1 = a(100)^3 + b(100)^2 + c(100)$, therefore,

$$1000000a + 10000b + 100c = 2760.1$$

$$\begin{bmatrix} 27 & 9 & 3 & 70.05 \\ 4096 & 256 & 16 & 359.98 \\ 1000000 & 10000 & 100 & 2760.1 \end{bmatrix} \cdots \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0.0013 \\ 0 & 1 & 0 & -0.09 \\ 0 & 0 & 1 & 23.61 \end{bmatrix}$$

Solution: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.0013 \\ -0.09 \\ 23.61 \end{bmatrix}$

a. The cubic polynomial that fits the data is

$$V(t) = 0.0013 \cdot t^3 - 0.09 \cdot t^2 + 23.61 \cdot t \qquad 0 \le t \le 126$$

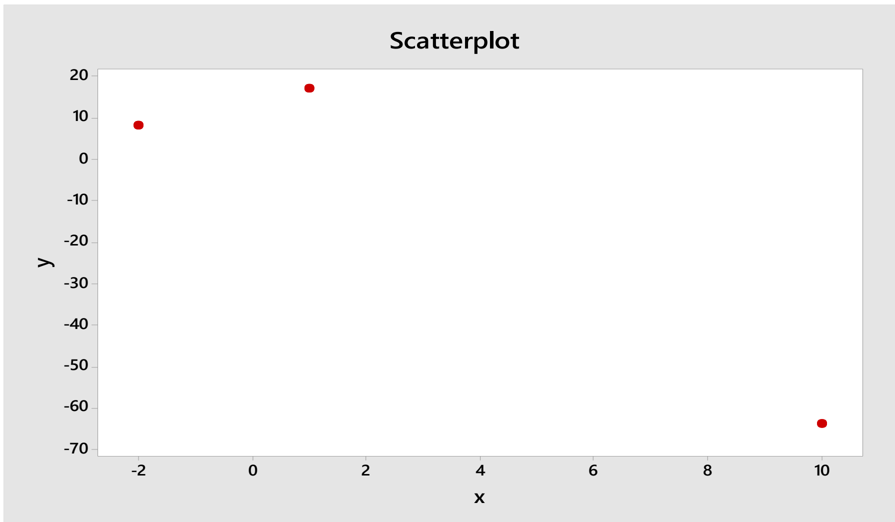
b. $V(126) = 0.0013 \cdot (126)^3 - 0.09 \cdot (126)^2 + 23.61 \cdot (126) \simeq 4{,}147 \ ft/s$

(approximately 2,827 mph)

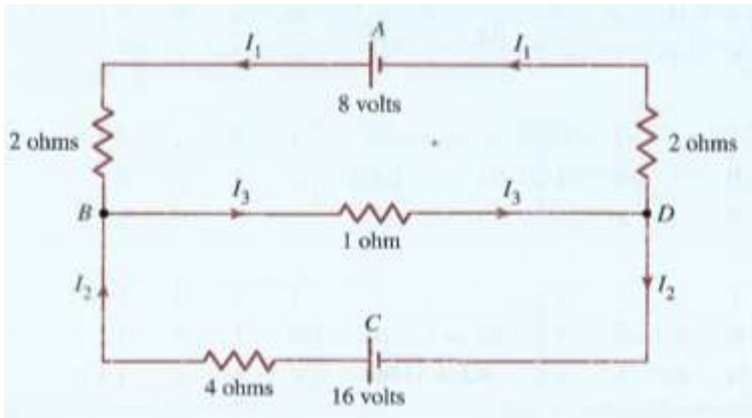
Homework

1. Find the unique parabola $\left(y = ax^2 + bx + c \right)$ that fits the following data points.

$$(1, 17), \quad (-2, 8) \quad \text{and} \quad (10, -64)$$



2. Find the currents in the given circuit.



Kirchhoff's Laws are used to set up all necessary equations to determine these currents.

Current Law: All currents flowing into a junction must flow out of it.

Voltage Law: The sum of all voltage drops $(V = IR)$ in a closed loop (any direction) is equal to the total voltage in that direction.

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 4I_1 + I_3 &= 8 \\ 4I_2 + I_3 &= 16 \end{aligned}$$